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An embedding technique for the solution of coupled Riccati equations

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Abstract. A new method of solving analytically coupled Riccati equations by means of embedding them into a matrix Riccati equation is proposed. This process imposes certain conditions onto the coefficients of the original equations. Whenever it is impossible to match these requirements, the calculation of less demanding invariants is discussed. The algebraic structure of these is simple enough to be practically used for lowering the system's dimension. In general, the newly soluble equations do not possess the Painlevé property.

1. Introduction

Systems of ordinary differential equations play a crucial role in nonlinear science. Unfortunately, analytical solution algorithms are only available for quite special classes (Cairó *et al* 1989, Goriely and Brenig 1990, Kamke 1967, Ramani *et al* 1989). Thus, the aim of this paper is to add to the arsenal of methods for obtaining closed analytical solutions (cf also Escher 1980, 1981, Levine and Tabor 1988). It is also intended to extend the structure of time-dependent integrals over the Carleman-type ones (cf Cairó *et al* 1989, Levine and Tabor 1988, Ramani *et al* 1989).

When discussing special cases, we have in mind (generalized) Lotka–Volterra equations: $f_{ijj}=0$ for $j\neq i$ in (1) below (cf Cairó *et al* 1989) and their applications, e.g. as simple semiconductor laser rate equations (Petermann 1988), in semiconductor recombination models (Schöll 1982), and in the form $\dot{u}=u(2u-v)$, $\dot{v}=v(u+1+v/m)$ arising in the theory of the porous media equation (Dresner 1983, 1990). The restriction $f_{ijj}=0$ for $j\neq i$ is less restrictive than it may seem at first glance (cf Ramani *et al* 1989), but some of the variety of complex behaviour, such as several limit cycles even for $N=2$ and time-independent coefficients (Escher 1981), may be absent.

When the nonlinear terms are all of second order,

$$\dot{x}_i=d_i+\sum_{j=1}^{N} e_{ij}x_j+\sum_{j,k=1}^{N} f_{ijk}x_jx_k \qquad i=1,2,\ldots,N \tag{1}$$

($\dot{x}\equiv \mathrm{d}x/\mathrm{d}t$), such systems may be called coupled Riccati equations. Analytical solutions are known only in some special cases, e.g. if $f_{ijk}=f_j\delta_{ik}$ (projective Riccatis). On the other hand, it is well known that the $N\times N$ *matrix* Riccati equation

$$\dot{\boldsymbol{x}}=\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{x}+\boldsymbol{x}\cdot\boldsymbol{b}'+\boldsymbol{x}\cdot\boldsymbol{c}\cdot\boldsymbol{x} \tag{2}$$

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is linearizable due to the non-commutative property of matrices. In this paper we investigate situations where system (1) can be represented in the matrix form (2). Because system (2) contains N^2 equations compared with N ones in (1), this representation is called an imbedding. Initially we consider the general case and concentrate then on that of two dependent variables, $N=2$ (sections 2 and 3, respectively). Whenever the compatibility conditions required for a complete solution in terms of (2) are too rigid for the system under consideration, one may try to compute first integrals (invariants), in order to reduce the dimensionality of the problem. This is examined in section 4. Section 5 finally discuss our results, including their relation to other soluble cases known from the literature.

2. The embedding method

There are $N^2(N+1)/2$ coefficients $f_{ijk}(=f_{ikj})$ in (1), but only N^2 matrix elements of c. Hence, one has to look for flexibility in the transition from (1) to (2). In order that (2) represents (1), we thus make the ansatz

$$\boldsymbol{x}(t)=\sum_{i=1}^{N} \boldsymbol{A}_i(t)x_i(t)+\boldsymbol{A}_0(t) \tag{3}$$

where the $N\times N$ matrices $\boldsymbol{A}_0 \ldots \boldsymbol{A}_N$ are also to be calculated.

By inserting the ansatz (3) into (2), using (1), one gets the *compatibility relations*

$$\dot{\boldsymbol{A}}_0+\sum_i \boldsymbol{A}_i d_i=\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{A}_0+\boldsymbol{A}_0\boldsymbol{b}'+\boldsymbol{A}_0\boldsymbol{c}\cdot\boldsymbol{A}_0 \tag{4a}$$

$$\dot{\boldsymbol{A}}_i+\sum_j \boldsymbol{A}_j e_{ji}=(\boldsymbol{b}+\boldsymbol{A}_0\boldsymbol{c})\boldsymbol{A}_i+\boldsymbol{A}_i(\boldsymbol{b}'+\boldsymbol{c}\cdot\boldsymbol{A}_0) \qquad i=1\ldots N \tag{4b}$$

$$2\sum_i \boldsymbol{A}_i f_{ijk}=\boldsymbol{A}_j\boldsymbol{c}\cdot\boldsymbol{A}_k+\boldsymbol{A}_k\boldsymbol{c}\cdot\boldsymbol{A}_j \qquad j,k=1\ldots N. \tag{4c}$$

In spite of the need to determine the unknowns $\boldsymbol{A}_0\ldots\boldsymbol{A}_N$, these equations represent necessary and sufficient conditions for the coefficients of (1). Only (4a) is a trivial equation for $\boldsymbol{a}$, whereas those in (4c) are $N(N+1)/2$ nonlinear matrix equations for the $N+1$ matrices $\boldsymbol{c}$ and $\boldsymbol{A}_1\ldots\boldsymbol{A}_N$, and (4$b$) are then N linear matrix equations for the three matrices $\boldsymbol{A}_0$, $\boldsymbol{b}$ and $\boldsymbol{b}'$.

Although the linear terms in (1) can also play a decisive role for the integrability (cf Ramani *et al* 1989), we have found it more complicated to match the conditions (4c) for the physically relevant systems mentioned above and, thus, will concentrate mainly on them.

If, for instance, $\boldsymbol{A}_1$ is assumed to be invertible, one obtains immediately

$$\boldsymbol{c}=\boldsymbol{A}_1^{-1}\sum_i \boldsymbol{A}_i f_{i11}\boldsymbol{A}_1^{-1}\equiv\tilde{\boldsymbol{c}}\cdot\boldsymbol{A}_1^{-1}. \tag{5}$$

Defining

$$\boldsymbol{B}_i=\boldsymbol{A}_1^{-1}\boldsymbol{A}_i \qquad i=0\ldots N \tag{6}$$

it is seen that $\boldsymbol{A}_1$ drops out and there remain only N free matrices to fulfil (4c). Furthermore,

$$\boldsymbol{b}'=\boldsymbol{A}_1^{-1}\dot{\boldsymbol{A}}_1+\sum_i \boldsymbol{B}_i e_{i1}-\boldsymbol{A}_1^{-1}\boldsymbol{b}\cdot\boldsymbol{A}_1-\boldsymbol{B}_0\tilde{\boldsymbol{c}}-\tilde{\boldsymbol{c}}\cdot\boldsymbol{B}_0. \tag{7}$$

In general, the more the compatibility relations (4) impose conditions onto the coefficients of (1), the higher N is. There are $N^2(3N+1)/2$ equations for the $N^2(N+4)$ transformation matrix elements $(A_0)_{ij} \ldots c_{ij}$, thus about $N^2(N-3)/2$ conditions for the $N^2(N+1)/2$ f_{ijk}s and N^2 e_{ij}s; the nonlinear character of (4*b*) and (4*c*) prevents a simple counting of free parameters, however.

We now examine the situation in more detail for the case $N=2$.

3. The case $N=2$

3.1. *Regular matrix A_i*

Equations (4*c*) yield (omitting the unit matrix, I, as an obvious factor)

$$\tilde{c}=f_{111}+B_2f_{211} \tag{8}$$

$$B_2^2f_{211}+B_2(f_{111}-f_{212})-f_{112}=0 \tag{9a}$$

$$B_2^2f_{212}+B_2(f_{112}-f_{222})-f_{122}=0. \tag{9b}$$

There are two principally different cases, not solvable ones like $f_{211}=0, f_{111}=f_{212}$, but $f_{112}\neq 0$ can be circumvented by a linear transformation of the dependent variables, x_i:

(i) B_2 is proportional to the unit matrix, $B_2=B_2I$. We remark at once, however, that then $x_1(t)$ and $x_2(t)$ cannot be extracted separately from the solution of (2) as

$$x=A_1(x_1+B_2x_2)+A_0 \tag{10}$$

in this case. By virtue of (9), *one* condition is imposed on the f_{ijk}s. This and similar cases are discussed in section 4.

(ii) B_2 is a solution of (9*a*) and (9*b*) with $f_{211}f_{212}\neq 0$ and is not proportional to I. Then the elimination of B_2^2 and comparison of coefficients gives the *two* conditions

$$f_{112}f_{212}=f_{122}f_{211} \tag{11a}$$

$$f_{212}(f_{111}-f_{212})=f_{211}(f_{112}-f_{222}). \tag{11b}$$

Further, in order to satisfy (4*b*), one may chose A_i to be time independent and $b=0$ (this would simplify the solution of the linearized equations) yielding a linear system of equations for the elements of B_0,

$$(B_0B_2-B_2B_0)\tilde{c}=e_{12}+B_2(e_{22}-e_{11}-B_2e_{21}) \tag{12}$$

(B_2 and $\tilde{c}$ commute), and from (7)

$$b'=e_{11}+B_2e_{21}-B_0\tilde{c}-\tilde{c}\cdot B_0. \tag{13}$$

If the LHS of (12) vanishes for some reason, insertion of (9*a*) into the RHS yields by virtue of the linear independence of B_2 and B_2^2 *two* conditions, namely

$$e_{12}f_{211}=e_{21}f_{112} \tag{14a}$$

$$(e_{22}-e_{11})f_{211}=e_{21}(f_{212}-f_{111}). \tag{14b}$$

These conditions can be circumvented by choosing $b\neq 0$, however. The conditions (11) turn out to be very severe. For this reason one could consider the case of singular matrices A_i.

3.2. *Singular matrices A_i*

Equations (4*c*) read

$$A_1 f_{111} + A_2 f_{211} = A_1 c \cdot A_1 \tag{15a}$$

$$2A_1 f_{112} + 2A_2 f_{212} = A_1 c \cdot A_2 + A_2 c \cdot A_1 \tag{15b}$$

$$A_1 f_{122} + A_2 f_{222} = A_2 c \cdot A_2. \tag{15c}$$

Now, det $A_i = 0$ implies the projector properties

$$A_i^2 = \mathrm{Tr}(A_i) A_i \tag{16a}$$

$$A_i c \cdot A_i = c_{A_i} A_i \qquad c_{A_i} \equiv c_{11} a_{11}^{(i)} + c_{12} a_{21}^{(i)} + c_{21} a_{12}^{(i)} + c_{22} a_{22}^{(i)}. \tag{16b}$$

In view of (15*a*) and (15*c*), linear independence of A_1 and A_2 requires immediately that

$$f_{211} = f_{122} = 0. \tag{17}$$

This suggests this case to be particularly suitable for Lotka–Volterra systems. The matrix c is then a *generalized inverse* (Pringle and Rayner 1971) of *both* A_1 and A_2 and contains a number of *arbitrary* matrix elements. It is hoped, consequently, that these degrees of freedom yield additional flexibility.

For $N = 2$, however, a careful treatment of the system (15) only leads to the solution of the projective Riccati case.

3.3. *Mixed case: A_1 regular, A_2 singular*

Repeating the corresponding operations of the foregoing subsections 3.1 and 3.2, one obtains the following:

$$f_{112} = f_{122} = 0 \tag{18a}$$

$$f_{212} - f_{111} = f_{211} a_{A_2}^{(-1)} \tag{18b}$$

$$f_{222} = c_{A_2} = a_{A_2}^{(-1)} (f_{111} + f_{211} a_{A_2}^{(-1)}) \tag{18c}$$

where $a_{A_2}^{(-1)}$ belongs to the inverse, A_1^{-1}, of matrix A_1, i.e.

$$a_{A_2}^{(-1)} \equiv a_{11}^{(-1)} a_{11}^{(2)} + a_{12}^{(-1)} a_{21}^{(2)} + a_{21}^{(-1)} a_{12}^{(2)} + a_{22}^{(-1)} a_{22}^{(2)} \tag{19}$$

and c_{A_2} is defined in (16*b*).

Thus, in dependence on the vanishing of the coefficient f_{211} there are two main cases:

Case (i), $f_{211} \neq 0$. Equations (18) then yield

$$f_{222} = c_{A_2} \tag{20a}$$

$$a_{A_2}^{(1)} = (f_{212} - f_{111}) / f_{211} \tag{20b}$$

$$f_{222} f_{211} = (f_{212} - f_{111}) f_{212}. \tag{20c}$$

Case (ii), $f_{211} = 0$. Here,

$$f_{222} = c_{A_2} \tag{21a}$$

$$a_{A_2}^{(1)} = f_{222} / f_{111} \tag{21b}$$

$$f_{111} = f_{212}. \tag{21c}$$

Again, although there are more adjustable matrix elements, $a_{ij}^{(1,2)}$, c_{ij}, than conditions to be fulfilled, the nonlinear character of these conditions causes relationships between the coefficients f_{ijk} to be satisfied, too. In return, the elements of A_i and c are not completely determined.

4. Invariants

The last section has resulted in compatibility relations (see (4)) between systems (1) and (2) which impose certain conditions onto the coefficients in (1). These conditions are likely to prevent practical applications. At the same time the number of these conditions is diminished if some of the matrices A_i are allowed to be proportional to each other. Thus, a solution is not obtained for each dependent variable, $x_i(t)$, separately, but only for a linear combination of at least two of them representing, thus, a first integral or, after elementary operations, a (time-dependent) invariant. In the simplest case,

$$\boldsymbol{A}_i = B_i\boldsymbol{A}_1 = B_i\boldsymbol{I} \qquad B_1 = 1 \qquad i = 1, 2, \ldots, N \tag{22}$$

Equations (4c) represent, then, one equation for $\boldsymbol{c}$ (becoming diagonal as well) and a system of $(N/2)(N+1)-1$ quadratic equations for the $N-1$ numbers $B_2 \ldots B_N$. Similarly, (4b) then represent one equation for $\boldsymbol{b}$ (or $\boldsymbol{b}'$; $\boldsymbol{A}_0 = \boldsymbol{0}$ without loss of generality) and $N-1$ conditiosn onto the N^2 coefficients e_{ij}. Both are likely to reduce the number of conditions onto the coefficients f_{ijk} and e_{ij} by a factor of the order of N. In other words, in such cases, there is a linear combination $z = \Sigma_i B_i x_i$ which obeys a single Riccati equation.

Moreover, if there are M linearily independent solutions $\{B_i\}$ of the system (4c) for which (4b) can be fulfilled, too, then the (time-dependent) invariant

$$\left(a_{11}^{(0)} + a_{11}^{(1)}\ {}^3\sum_{i=1}^{N} B_i x_i(t)\right) \Big/ x_{11}(t) = 1 \tag{23}$$

represents actually M different first integrals.

The case $N = 2$ is trivial, of course, so we give few formulae for illustrational purposes. Let $\boldsymbol{A}_2 = B_2\boldsymbol{A}_1$ (but not necessarily $\boldsymbol{A}_1$ be regular). B_2 obeys (9a) and (9b) simultaneously, where it replaces $\boldsymbol{B}_2$. One obtains *one* compatibility condition between the coefficients f_{ijk}, the concrete form of which depends on which f_{ijk} are different from zero.

For instance, if $f_{ijj} = 0$ for $i \neq j$, then

$$B_2 = f_{222}/f_{111} \tag{24}$$

and the compatibility relation reads

$$(f_{222} - f_{112})f_{111} = f_{222}f_{212}. \tag{25}$$

Further, for regular $\boldsymbol{A}_i$ it follows from (4b) that

$$e_{11} + B_2 e_{21} = \boldsymbol{b} + \boldsymbol{A}_0\boldsymbol{c} + \boldsymbol{b}' + \boldsymbol{c} \cdot \boldsymbol{A}_0 \tag{26a}$$

$$e_{12} + B_2 e_{22} = B_2(e_{11} + B_2 e_{21}) \tag{26b}$$

and (26a) determines, for example $\boldsymbol{b} = (e_{11} + B_2 e_{21})\boldsymbol{I}$, when for simplicity $b' = 0$, $\boldsymbol{A}_0 = 0$.

In any case, there is a linear combination $z = B_1 x_1 + B_2 x_2$ which obeys a single Riccati equation, $\dot{z} = Cz^2 + Dz + E$, from which all relations can be derived by comparison of coefficients.

5. Discussion

A new method of analytically solving coupled Riccati equations is proposed. Its application requires certain interrelations of coefficients to be satisfied. Not surprisingly (cf Ramani *et al* 1989), the conditions for the coefficients of the nonlinear terms f_{ijk} turn out to be the most restrictive ones. The families of equations solvable by this method comprise, at least for two dependent variables, the projective Riccati case (Jacobi equation) $f_{122}=f_{211}=0$, $2f_{212}=f_{111}$, $f_{222}=2f_{212}$.

The range of applicability can be extended by means of transformations of the original equations. In particular, generalized Lotka–Volterra equations of the type

$$\dot{x}_i = \lambda_i x_i + x_i \sum_j A_{ij} \prod_k x_k^{B_{jk}} \tag{27}$$

can be brought into the form (1) (Goriely and Brenig 1990).

Ramani *et al* (1984) have performed a rigorous Painlevé analysis of the system (1) with $N=2, f_{122}=f_{211}=0, f_{111}=f_{222}=-1, 2f_{112}=a, 2f_{212}=b$. The present method yields immediately a first integral for $a+b=-2$ and $\alpha+\gamma=\beta+\delta$ (i.e. $e_{11}+e_{21}=e_{12}+e_{22}$). Indeed, then the sum $z=x+y$ satisfies a single Riccati equation. After a linear transformation of the dependent variables, case (i) in subsection 3.3 provides explicit solutions for various combinations of values for a and b. One example is $a=0$ ($b=0$), but *no* restriction for $b(a)$ (the conditions for the other coefficients are to be examined separately). Consequently, there are other whole families of coefficients for which system (1) is integrable, although it does not pass the Painlevé test. While the Painlevé analysis picks out *entire* values for *a and b*, the present results, like others (e.g. system (2) in Ramani *et al* (1984)), suggest that many integrable cases are not confined to entire-valued coefficients. In other words, the Painlevé property is sufficient for integrability, but well beyond necessity.

On the other hand, in all known cases of integrability of (1) certain relations between its coefficients *are* obeyed. It is thus suggested that integrability could be related to some modified Painlevé property in that the only movable singularities are not poles for *each* dependent variable, but branch points with certain relations between the exponents. This property would be even 'weaker' than 'weak Painlevé' (Tabor 1989), but, of course, less general than the psi-series (Levine and Tabor 1988).

According to the form of the solutions of (2), integrals of motion of (1) calculated by the proposed method would be of the form (linear combination of x_is) $\times T(t)$ with $T(t)=\Sigma_i p_i \exp(\lambda_i t)/\Sigma_j q_j \exp(\lambda_j t)$, where the λ_is are eigenvalues of matrices derived from (2) and at once linear combinations of the eigenvalues of the corresponding Jacobian of (1). This form is distinctly simpler than the structure (algebraic function of the x_is) $\times\, e^{\delta t}$ evolving from the Carleman embedding method, so that it is hoped to get not only new invariants for new parameter families, but also to benefit from the simple form of these invariants for the practical reduction of the system's dimension.

In summary, the embedding method proposed here solves new families of coupled Riccati equations. Its full range of applicability remains to be investigated.

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